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Lateral Conditions for Semigroups Involving Mappings in L^p. I

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INTRODUCTION

Let X be an ordered locally convex topological vector space and suppose $A : D \to X$ is a linear operator with the following property: there exists a contraction A_0 of A such that A_0 is the infinitesimal generator of an equicontinuous semigroup $\{T_t^0\}$ of positive operators which is strongly continuous for $t \geqslant 0$ on a subspace $X_0 \subset X$, i.e., $\{T_t^0\}$ is an equicontinuous semigroup of class C_0 on X_0. We refer to Yosida [9], p. 234 ff. for the definition and properties of such semigroups. Let the resolvent $[\lambda I - A_0]^{-1}$ be denoted by R_λ^0. Even though X_0 may be a proper subspace of X, it is assumed that R_λ^0 has an extension to the entire space. See section 1(c) for the details. In this paper we shall make a study of the resolvents R_λ which dominate R_λ^0 and correspond to equicontinuous semigroups of class C_0 on subspaces of X, with infinitesimal generators which are contractions of A.

The case in which the null space of A is finite dimensional has been studied in [3]. In the present paper we shall remove that restriction and extend our previous results to a more general setting. Many of the most interesting examples involve operators A with infinite dimensional null spaces, the obvious one being that of a differential operator such as the Laplacian in $n \geqslant 2$ dimensions. Another example arises from the Kolmogorov differential equations where A is an infinite matrix

$$A = -q + q\Pi$$

with q a positive diagonal matrix and Π a substochastic matrix. We discussed this example in [3], but were forced there to assume that A had finite dimensional null space. In general this condition is not satisfied, so that one is led even in this example to consider an infinite dimensional null space. Other examples are afforded by integrodifferential operators arising in the theory of the symmetric stable processes with parameter α, $0 < \alpha < 2$. These will be discussed in a later paper.

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In our treatment the emphasis is on "lateral conditions," a notion first introduced by W. Feller, which generalize the classical boundary conditions for differential operators. We show that each of the dominating resolvents corresponds to a lateral condition. This lateral condition defines the contraction of A that generates the corresponding semigroup. See Theorem 4.1, where the lateral condition is given by formula (4.8). Conversely, given any one of a certain class of lateral conditions, we can construct a corresponding semigroup whose infinitesimal generator is the contraction of A obtained by restricting the domain to those elements satisfying the given lateral condition [Theorem 7.1].

The original paper which led us to the present problem was that of Feller [4] in which he studied the lateral conditions for the Kolmogorov differential equations. Feller's work has been extended by Chung [1], Williams [8], and Dynkin [2]. For a detailed probabilistic analysis of the results, see especially [1]. Another interesting approach to the same sort of problem we consider here has recently been made by M. Fukushima [5] in the case that $A = \frac{1}{2}\Delta$ ($\Delta =$ Laplacian) on a bounded domain D in R^n, $n \geqslant 2$, with $R_\lambda{}^0$ the minimal resolvent corresponding probabilistically to the absorbing barrier Brownian motion process on D. He uses the theory of Dirichlet spaces, due to Beurling and Deny, to find a certain class of symmetric resolvents dominating $R_\lambda{}^0$.

In Section 1 we give the basic assumptions about the underlying space, the given resolvent, and the null space of A. In Section 2 we show that a large class of dominating resolvents can be put into a certain standard form (Lemma 2.1) involving what we term "admissible families of mappings" (Definition 2.1). In Section 3 we collect some of the most useful properties of admissible families. In Section 4 we show that if R_λ is a dominating resolvent then each $F \in$ range R_λ must satisfy a "lateral condition" (Theorem 4.1). Section 5 is devoted to finding a specific formula for the term that must be added on to the given resolvent $R_\lambda{}^0$ to get the dominating resolvent R_λ. Finally, in Sections 6 and 7 we assume that the lateral conditions are given. We then construct dominating resolvents (Section 6) and eventually semigroups (Section 7) generated by restrictions of A to the linear spaces defined by the lateral conditions.

Finally, we should like to discuss briefly two restrictions that are made in this paper. The first is the assumption first encountered in Section 4, formula (4.6). If this condition is not satisfied then a different train of arguments is required to obtain the analogous of Theorem 4.1 and the theorem of Sections 5-7. Condition (4.6) is analogous to the condition in [3] or [4] that the matrix V_{ij}^λ increases as $\lambda \to \infty$ to a finite limiting matrix V_{ij}. We intend to treat in a future paper the case where (4.6) does not hold.

The second restriction we referred to is on the mapping S in (6.2). We assume that $[I - S]^{-1}$ exists, but this is not a necessary condition. It is possible that the general situation could be obtained from our results as a limiting case.

A detailed treatment of examples will be deferred to Part II of this paper.

1. Preliminaries

We shall be working in the framework of a locally convex linear topological space X which is an ordered topological vector space over R in the sense of [7], Chapter V, that is, X has an order structure defined by a reflexive, transitive, and antisymmetric binary relation $\leqslant$ such that

$$x \leqslant y \to x + z \leqslant y + z \qquad \forall x, y, z \in X \tag{1.1}$$

$$x \leqslant y \to cx \leqslant cy \qquad \forall x, y \in X, \qquad \forall c > 0. \tag{1.2}$$

and the positive cone C satisfies:

$$C = \{x : x \geqslant 0\} \qquad \text{is closed in} \qquad X. \tag{1.3}$$

(a) The Given Operator. We assume given a densely defined linear operator A with values in X. We call its null space N, and let $N^+ = C \cap N$, where C is the positive cone (1.3). We shall impose the following conditions on N^+:

N_1: N^+ is closed.

N_2: N^+ is a lattice.

N_3: N^+ has a metrizable universal cap.

Recall that a nonempty subset G of a closed convex cone is called a cap if G is compact, convex, and its complement in the cone is convex. The cap is called universal if the cone is equal to $\bigcup_{n=1}^{\infty} nG$. We refer to Phelps [6], chapter 11 for the basic theorems concerning caps. The condition N_3 is somewhat more general than the assumption of a compact base for N^+. It should be noted that the condition does not imply the metrizability of N^+ [see [6], the example on p. 91]. Let us sum up briefly the properties that will be important for our purposes. First of all, since N^+ is a lattice, any cap of N^+ is a Choquet simplex, cf. [6], Proposition 11.3, p. 93. Let us denote the universal cap again by G. There exists a lower semicontinuous finite valued function $p \geqslant 0$ on N^+ which is additive and positive homogeneous such that $G = \{x \in N^+ : p(x) \leqslant 1\}$. Furthermore, the nonzero extreme points of G are contained in the set $G_1 = \{x \in N^+ : p(x) = 1\}$. The set G_1 may not be compact and so is not necessarily a base of N^+. Thus we have here a more general situation than the case of a compact base. See [6], pp. 89-90 for a proof of these statements. Since G is a metrizable simplex the extreme points form a G_δ and any point in G is represented by a unique maximal measure supported by the extreme points of G. If $x \in G_1$, then this maximal measure is supported by the nonzero extreme points. Since any nonzero point of the cone N^+ is a multiple of an element of G_1, we get an integral representation for elements in N^+ using the Choquet representation theorem. The Choquet theorem ([6], p. 19) tells us that each $x \in G$ is represented by a unique probability measure μ supprted by the extreme points of G, that is

$$\int_{\partial_e G} y\mu_x(dy) = x \tag{1.4}$$

where $\partial_e G =$ the extreme points of G and $\mu_x(\partial_e G) = 1$. If x is a non-zero element of N^+ then we can replace $\partial_e G$ with $\partial_e G \setminus \{0\} \subset G_1$ in (1.4), cf. [6], p. 90.

(b) Reference Measures on the Extreme Boundary. Let

$$B = \partial_e G \setminus \{0\}$$

and let m be a fixed positive regular Borel measure on the compact set $\bar{B}$, concentrated on B. Denote by N_m^+ the set of elements in N^+ whose representing measures are absolutely continuous with respect to m, i.e., x is of the form

$$x = \int_B \phi_x(y)\, ym(dy) \tag{1.5}$$

where $\phi_x \in L_m{}^1(B)$ with $\phi_x \geqslant 0$. We shall use the notation

$$N_m = N_m^+ - N_m^+ .$$

From now on we shall assume a fixed but arbitrary reference measure m satisfying the above conditions.

For each q such that $1 \leqslant q \leqslant \infty$ we define $N_m{}^q$ to be the subspace of N_m such that the ϕ_x in (1.5) belongs to $L_m{}^q$.

(c) The Given Resolvent. We suppose given a family $\{R_\lambda{}^0;\ \lambda > 0\}$ of linear transformations with the following properties:

(i) $R_\lambda{}^0$ is defined on a subspace W of X which contains N_m. In addition, there exists a closed subspace X_0 of X such that for some q

$$\overline{R_\lambda{}^0(X_0 + N_m{}^q)} = X_0 \tag{1.6}$$

and $R_\lambda{}^0$ is continuous on $X_0 + N_m{}^q$ for each $\lambda > 0$. Here the sum $X_0 + N_m{}^q$ is a vector sum, not a topological direct sum.

(ii) If $\lambda > 0$ and $f \in X_0 + N_m$, then $R_\lambda{}^0 f$ is the unique solution in X_0 to the equation $\lambda u - Au = f$. This uniqueness condition implies the resolvent equation

$$R_\lambda{}^0 f - R_\mu{}^0 f = (\mu - \lambda)\, R_\lambda{}^0 R_\mu{}^0 f \qquad (\lambda, \mu > 0). \tag{1.7}$$

We shall suppose that every element $f \in X_0 + N_m$ can be expressed as a difference $f = f_1 - f_2$, where $f_i \in C_m = C \cap (X_0 + N_m)$, i.e., the positive cone C_m is generating in $X_0 + N_m$.

(d) The Null Space of $\lambda I - A$. Given $\lambda > 0$, let

$$N_\lambda = \{u : \lambda u - Au = 0\}$$

$$N_\lambda^+ = C \cap N_\lambda .$$

If $x \in N_m$ define

$$h^\lambda(x) = x - \lambda R_\lambda{}^0 x. \tag{1.8}$$

Then $h^\lambda(x) \in N_\lambda$. We shall make the important assumption that the mapping h^λ is order preserving, i.e., $h^\lambda N_m^+ \subset N_\lambda^+$.

The mapping h^λ may not be $1 - 1$. However, by the resolvent equation (1.4) it follows that if $h^\lambda(x) = 0$ then $h^\mu(x) = 0$, $\forall \mu > 0$. We have then the following definition:

Definition 1.1. *The zeros of h^λ are called passive solutions of $Ax = 0$.*

The passive solutions of $Ax = 0$ which lie in G form a closed subset of G, since they are just the zeros of the continuous linear transformation $R_\lambda{}^0$ restricted to the closed subspace X_0. Let B_p denote the elements of $\partial_e G \setminus \{0\}$ which are passive and let $B_a = B \setminus B_p$.

Now N_m is the (algebraic) direct sum of two subspaces $N_m{}^p$ and $N_m{}^a$ such that $N_m{}^p \cap N_m{}^a = \{0\}$. If $x \in N_m$ we have

$$x = \int_{B_p} y\mu_x(y)\, m(dy) + \int_{B_a} y\mu_x(y)\, m(dy).$$

Note that $N_m{}^a \cap X_0 = \{0\}$ since

$$x \in N_m{}^a \cap X_0 \rightarrow h^\lambda(x) \in X_0 \rightarrow h^\lambda(x) = 0 \rightarrow x \in N_m{}^p ,$$

using the uniqueness assumption in (c) part (ii).

(e) The Space $X_m{}^q$. We introduce a new topology on $X_0 + N_m{}^q$ which is in general stronger than the original one induced by X. The reason for doing this is that $X_0 + N_m{}^q$ may not be a closed subspace of X in the original topology and therefore this induced topology is not a very convenient one to work with. Given $q \geqslant 1$ we assume $\mu_x \in L_m{}^q$.

To define the new topology, we can express $X_0 + N_m$ as an algebraic direct sum $X_0 + N_m{}^a$. Let $\mathscr{P}$ denote a family of seminorms determining the topology of X. Now if

$$x = x_0 + \int_{B_a} y\mu_x(y)\, m(dy),$$

we define for each $q \geqslant 1$ a new family $\mathscr{P}'(q)$ of seminorms by

$$p_q'(x) = p(x) + \| \mu_x \|_{L^q} \tag{1.9}$$

where $p \in \mathscr{P}$. We shall require from now on that

$$x \geqslant 0 \to \mu_x(y) \geqslant 0 \qquad m - \text{a.e.} \qquad \text{on} \qquad B_a \,. \tag{1.10}$$

Note that if $\mathscr{P}$ is a system of monotone seminorms, then $\mathscr{P}'(q)$ is again a system of monotone seminorms for each q. Let $X_m{}^q$ be the space $X_0 + N_m{}^a$ with the topology generated by $\mathscr{P}'(q)$. The new topology then makes $X_m{}^q$ the topological direct sum of X_0 and $N_m{}^a$, since on X_0 the old and new topologies agree, and since on $N_m{}^a$ we have $p(x) \leqslant \text{const.} \| \mu_x \|_{L^q}$, using the compactness of $\bar{B}_a$.

Since $R_\lambda{}^0 X_m{}^q \subset X_0$ we conclude that $R_\lambda{}^0$ remains continuous on each of the spaces $X_m{}^q$.

2. Representation of the Dominating Resolvents

Given a fixed q such that $1 \leqslant q \leqslant +\infty$, we shall be concerned with positive semigroups on $X_m{}^q$ whose resolvents are of the form:

$$R_\lambda f = R_\lambda{}^0 f + h^\lambda \left\{ \int_{B_a} g^\lambda(f, y)\, y m(dy) \right\}, \tag{2.1}$$

where for each $f \in X_m{}^q$ we have $g^\lambda(f; \cdot) \in L_m{}^q(B_a)$ with m the fixed reference measure. For convenience of notation we shall often write $g^\lambda(f)$ instead of $g^\lambda(f; \cdot)$. Further, the mapping $f \to g^\lambda(f)$ is a positive, linear map from $X_m{}^q$ into $L_m{}^q(B_a)$.

From now on to avoid confusion between elements of the space $X_m{}^q$ and elements of $L_m{}^q(B_a)$, we shall in case of ambiguity denote the latter by gothic letters. We now introduce some mappings which occur frequently in the sequel. If

$$f = f_0 + \int_{B_a} \mathfrak{u}_f(y)\, y m(dy) \tag{2.2}$$

we define

$$z^a(f, S) = \int_{B_a \cap S} u_f \, dm \tag{2.3}$$

for each Borel set $S \subset B$. Then z^a is a positive continuous linear functional for each such S. If we further decompose

$$f_0 = f_1 + \int_{B_p} u_f(y)\, y m(dy) \tag{2.4}$$

we can define

$$z^p(f, S) = \int_{B_p \cap S} u_f(y)\, m(dy). \tag{2.5}$$

Again z^p is a linear functional for fixed S. In general it may not be a continuous one, but the continuity is not needed in what follows.

In terms of z^a we have in (2.1):

$$g^\lambda(f) = z^a(R_\lambda f), \tag{2.6}$$

where $z^a(F)$ denotes the $L_m{}^q$ density of the measure $z^a(F; \cdot)$.

Let us recall that a pseudo-resolvent R_λ defined for $\lambda > 0$ on $X_m{}^q$ is a family of continuous linear mappings from $X_m{}^q$ into itself such that the resolvent equation (1.7) holds (with $R_\lambda{}^0$ replaced by R_λ), cf. [9], p. 215 ff.

We shall find it convenient to use the abbreviation:

$$x_g = \int_B y g(y)\, m(dy) \tag{2.7}$$

where $g \in L_m{}^q(B)$. We can define a continuous positive linear mapping S from $L_m{}^q(B_a)$ into $L_m{}^q(B_a)$ by setting $g = 0$ on B_p and

$$S^\lambda g = \lambda g^\lambda(x_g). \tag{2.8}$$

This operator will play an important role in the theory. Another operator which will occur frequently is

$$T^\lambda g = \lim_{\mu \downarrow 0} \lambda g^\lambda[h^\mu(x_g)]. \tag{2.9}$$

Let us first verify that (2.9) makes sense for an appropriate mode of convergence. From the definition of h^μ in (1.8) we have

$$h^\mu(x) - h^\nu(x) = (\nu - \mu)\, R_\mu{}^0[h^\nu(x)]. \tag{2.10}$$

Thus for fixed $\lambda > 0$ and $g \geqslant 0$ we have

$$\| \lambda g^\lambda[h^\mu(x_g)] - \lambda g^\lambda[h^\nu(x_g)] \|_{L_m^q} = |\nu - \mu| \cdot \| \lambda g^\lambda(R_\mu{}^0[h^\nu(x_g)]) \|_{L_m^q} .$$

Thus (2.9) exists for each fixed g as an $L_m{}^q$ limit, i.e. if

$$T_\mu{}^\lambda g = \lambda g^\lambda[h^\mu(x_g)] \tag{2.11}$$

then $T_\mu{}^\lambda \to T^\lambda$ in the sense of strong operator convergence. It is clear that both $T_\mu{}^\lambda$ and T^λ are continuous linear operators from $L_m{}^q(B_a)$ into itself. It is also clear that

$$0 \leqslant T^\lambda \leqslant S^\lambda \tag{2.12}$$

in the usual ordering for operators.

LEMMA 2.1. *Suppose that R_λ is a pseudo-resolvent satisfying* (2.1) *for each $\lambda > 0$, where $g^\lambda(f, S)$ is a positive continuous linear functional on $X_m{}^q$ for each Borel set $S \subset B_a$. If*

$$\lim_{\mu\downarrow 0} g^\mu[(\mu I - A) R_\lambda{}^0 f] = d^\lambda(f) \tag{2.13}$$

exists for each fixed $f \in X_m{}^q$ and each $\lambda > 0$ as an $L_m{}^q(B_a)$ limit, then

$$g^\lambda(f) = [I - T^\lambda]\, d^\lambda(f) \tag{2.14}$$

where T^λ is given by (2.9).

PROOF. We have

$$g^\lambda[(\lambda I - A) R_\mu{}^0 u] = g^\lambda[(\lambda I - A) R_\mu u] - (\lambda - \mu)\, g^\lambda\left[h^\mu\left\{\int_{B_a} x g^\mu(u, x)\, m(dx)\right\}\right]. \tag{2.15}$$

By the resolvent equation and (2.6),

$$\begin{aligned} g^\lambda[(\lambda I - A) R_\mu u] &= g^\lambda[(\lambda - \mu) R_\mu u + u] \\ &= z^a[(\lambda - \mu) R_\lambda R_\mu u + R_\lambda u] = z^a[R_\mu u] = g^\mu(u). \end{aligned} \tag{2.16}$$

Thus (2.15) becomes

$$g^\lambda[(\lambda I - A) R_\mu{}^0 u] = g^\mu(u) - (\lambda - \mu)\, g^\lambda\left[h^\mu\left\{\int_{B_a} y g^\mu(u; y)\, m(dy)\right\}\right]. \tag{2.17}$$

Putting $u = (\mu I - A) R_\lambda{}^0 f$ in (2.17) we get

$$g^\lambda(f) = \left[I - \left(1 - \frac{\mu}{\lambda}\right) T_\mu{}^\lambda\right] d_\mu{}^\lambda(f), \tag{2.18}$$

where $T_\mu{}^\lambda$ is given by (2.11) and

$$d_\mu{}^\lambda(f) = g^\mu[(\mu I - A) R_\lambda{}^0 f]. \tag{2.19}$$

Thus

$$g^\lambda(f) - [I - T^\lambda]\, d^\lambda(f) = \left[T^\lambda - \left(1 - \frac{\mu}{\lambda}\right) T_\mu{}^\lambda\right] d^\lambda(f)$$

$$+ \left[I - \left(1 - \frac{\mu}{\lambda}\right) T_\mu{}^\lambda\right] [d_\mu{}^\lambda(f) - d^\lambda(f)]. \qquad (2.20)$$

The first term on the right tends to 0 with μ since $T_\mu{}^\lambda \to T$ strongly; the second term tends to 0 since $\| T_\mu{}^\lambda \| \leqslant \| S \|$ and $d_\mu{}^\lambda \to d^\lambda$ in $L_m{}^q$ by hypothesis.

The next lemma gives a sufficient condition for the existence of the limit (2.13).

LEMMA 2.2. *If for each fixed* $\mathfrak{f} \in L_m{}^q(B_a)$

$$(I - T^\lambda)^{-1}\, \mathfrak{f} = \lim_{N\to\infty} \sum_{n=0}^{N} (T^\lambda)^n\, \mathfrak{f} \in L_m{}^q \qquad (2.21)$$

exists as an $L_m{}^q$ *limit, then* (2.13) *holds and* $\int_S d^\lambda(f)\, dm$ *is a continuous linear functional on* $X_m{}^q$ *for each Borel set* $S \subset B_a$. *Also the* map $f \to d^\lambda(f)$ *from* $X_m{}^q \to L_m{}^q$ *is continuous.*

PROOF. Since $T_\mu{}^\lambda \leqslant T^\lambda$, if $\mu < \lambda$, it follows from (2.18) that if $f \geqslant 0$ in $X_m{}^q$

$$0 \leqslant \sum_{n=M}^{N} \left(1 - \frac{\mu}{\lambda}\right)^n (T_\mu{}^\lambda)^n g^\lambda(f) \leqslant \sum_{n=M}^{N} (T^\lambda)^n g^\lambda(f).$$

Hence for given $\epsilon > 0$ if $f \geqslant 0$ is fixed we can choose N_0 independent of μ such that

$$\left\| \sum_{n \geqslant N_0} \left(1 - \frac{\mu}{\lambda}\right)^n (T_\mu{}^\lambda)^n g^\lambda(f) \right\|_{L^q} < \epsilon.$$

We have also shown that $T_\mu{}^\lambda \to T^\lambda$ strongly as $\mu \downarrow 0$ for fixed $\lambda > 0$ and therefore $d_\mu{}^\lambda(f) \to d^\lambda(f) = (I - T^\lambda)^{-1} g^\lambda(f)$ in L^q for each fixed f.

COROLLARY 2.1. *If* (2.21) *holds with* T^λ *replaced by* S^λ *then* (2.13) *holds and the conclusion of Lemma 2.1 is valid.*

PROOF. This follows from $\| T^\lambda \| \leqslant \| S^\lambda \|$ and Lemma 2.2.

LEMMA 2.3. *Under the hypotheses of Lemma 2.1 the family* $\{d^\lambda(f);\, \lambda > 0\}$ *satisfies*

$$d^\lambda(f) - d^\mu(f) = (\mu - \lambda)\, d^\lambda(R_\mu{}^0 f) \qquad (2.22)$$

and

$$\lambda\, d^\lambda(f) - \mu\, d^\mu(f) = (\lambda - \mu)\, d^\lambda[-AR_\mu{}^0 f]$$
$$= (\lambda - \mu)\, d^\lambda[f - \mu R_\mu{}^0 f]. \qquad (2.23)$$

PROOF. This is an immediate consequence of the definition of d^λ in (2.13) and the resolvent equation.

We have thus shown that under certain conditions the representation (2.14) holds for $g^\lambda(f)$. This representation is not necessarily unique. There may be different families $\{d^\lambda\}$ satisfying (2.22) and operators T^λ giving such a representation. We shall show later that the particular one we have found here has a number of useful properties.

DEFINITION 2.1. *We shall call a family $\{d^\lambda(f); \lambda > 0\}$ of positive continuous linear mappings from $X_m{}^q$ into $L_m{}^q(B_a)$ and admissible family if* (2.22) *is satisfied.*

Let us close the section by comparing the situation so far to the case of the finite dimensional null space studied in [3]. There S^λ was given by a matrix

$$S^\lambda_{ij} = \lambda g_i{}^\lambda(x_j),$$

that is,

$$(S^\lambda \mathfrak{f})_i = \sum_{j=1}^{m} S^\lambda_{ij} \mathfrak{f}_j = \lambda g_i{}^\lambda(x_\mathfrak{f})$$

with x_j, $j = 1,..., m$ forming B_a. Here m is the measure giving mass 1 to each x_j, and we can take $q = +\infty$. If, for example $[S^\lambda_{ij}]$ is a substochastic matrix in the strict sense that

$$\sum_j S^\lambda_{ij} < 1$$

then $\| S^\lambda \| < 1$ as a map from L^∞_m into L^∞_m and so we have $[I - S^\lambda]^{-1}$ existing. This strong a condition is not necessary, however.

3. SOME PROPERTIES OF ADMISSIBLE FAMILIES OF MAPPINGS

Let us first note that formula (2.23) is a consequence of (2.22) so that any admissible family $\{d^\lambda\}$ automatically satisfies (2.23). From this the following lemma is immediate:

LEMMA 3.1. *If $\{d^\lambda; \lambda > 0\}$ is an admissible family of mappings, and if $0 \leqslant \lambda R_\lambda{}^0 f \leqslant f$ for all $\lambda > 0$ then $\lambda\, d^\lambda(f)$ increases as λ increases. If, in particular, $f = x$ is a passive solution of $Ax = 0$ then $\lambda\, d^\lambda(x)$ is constant in λ.*

We shall introduce two more operators. Recalling the notation introduced in formula (2.7) we define a mapping τ from $L_m{}^q(B_p)$ into $L_m{}^q(B_a)$ by setting $g = 0$ on B_a and

$$\tau g = \lambda d^\lambda(x_g). \tag{3.1}$$

Since x_g is passive the mapping τ is independent of λ. Next we define a map V^λ from $L_m{}^q(B_a)$ into $L_m{}^q(B_a)$ by

$$V^\lambda g = \lambda\, d^\lambda[x_g]. \tag{3.2}$$

By Lemma 3.1 we see that if $g \geqslant 0$ then $V^\lambda g$ increases with λ, or in the operator ordering V^λ increases with λ.

PROPOSITION 3.1. *If R_λ is of the form*

$$R_\lambda f = R_\lambda{}^0 f + h^\lambda \left\{ \int_{B_a} y g^\lambda(f, y)\, m(dy) \right\} \tag{3.3}$$

with

$$g^\lambda(f) = [I - T^\lambda]\, d^\lambda(f) \tag{3.4}$$

for some admissible family $\{d^\lambda; \lambda > 0\}$ then a linear, but not necessarily continuous, mapping d^0 from range AR_λ into $L_m{}^q(B_a)$ is defined by

$$d^0(-AR_\lambda f) = d^\lambda(f) - V^\lambda z^a(R^\lambda f) - \tau[z^p(R_\lambda f)]. \tag{3.5}$$

The representation is independent of λ in the sense that

$$-AR_\lambda f = -AR_\mu g \to d^0(-AR_\lambda f) = d^0(-AR_\mu g).$$

PROOF. Suppose X_0 is the algebraic direct sum of the subspaces X_1 and $N_m{}^p$. We first show that d^0 can be defined on the range of A/X_1 by the formula

$$d^0(-Ag) = d^\lambda(\lambda g - Ag). \tag{3.6}$$

Since X_1 contains no solutions of $Ax = 0$ we see that $Ag = 0$ and $g \in X_1$ implies $g = 0$. Hence for fixed λ a linear functional is certainly well defined by (3.6). To check that the definition is independent of λ we must show that

$$d^\lambda(\lambda g - Ag) = d^\mu(\mu g - Ag) \tag{3.7}$$

$\forall g \in$ domain $A \cap X_1$ and $\forall \lambda, \mu > 0$. Now by (2.22) and (2.23)

$$\begin{aligned} \lambda\, d^\lambda(g) - \mu\, d^\mu(g) &= (\lambda - \mu)\, d^\lambda(-AR_\mu{}^0 g) \\ &= (\lambda - \mu)\, d^\lambda(-R_\mu{}^0 A g) = d^\lambda(Ag) - d^\mu(Ag), \end{aligned} \tag{3.8}$$

which is just (3.7). Now let $f \in X_m{}^q$ and define

$$g = R_\lambda f - \int_{B_a} z^a(R_\lambda f; y)\, y m(dy) - \int_{B_p} z^p(R_\lambda f; y)\, y m(dy). \tag{3.9}$$

Then $g \in X_1$, and we get

$$d^0(-AR_\lambda f) = d^0(-Ag) = d^\lambda(\lambda g - Ag)$$
$$= d^\lambda(f) - V^\lambda z^a(R_\lambda f) - \tau z^p(R_\lambda f). \qquad (3.10)$$

Clearly if $R_{\lambda_1} f_1 = R_{\lambda_2} f_2$ then both of these representations lead to the same g in (3.9) and therefore $d^0(-AR_{\lambda_1} f_1) = d^0(-AR_{\lambda_2} f_2)$.

4. Lateral Conditions

Throughout the rest of the paper we assume $q \in [1, \infty]$ to be fixed.

Lemma 4.1. *If $R_\lambda \geqslant R_\lambda^0$ is a pseudo-resolvent of the form* (3.3)-(3.4) *for some admissible family then each $F \in$ range R_λ satisfies:*

$$[I - S^\lambda]\, z^a(F) = [I - T^\lambda]\, [d^0(-AF) + q(F)], \qquad (4.1)$$

where S^λ is defined by (2.8) *and*

$$q(F) = \tau[z^p(F)] \qquad (4.2)$$

with τ given by (3.1).

Proof. Suppose that for some $f \in X_m{}^q$ we have $F = R_\lambda f$. Then

$$z^a(F) = g^\lambda[\lambda F - AF] = [I - T^\lambda]\, d^\lambda(\lambda F - AF). \qquad (4.3)$$

By (3.5)

$$d^0(-AF) = d^\lambda(\lambda F - AF) - V^\lambda z^a(F) - q(F). \qquad (4.4)$$

Therefore, using the fact that

$$S^\lambda = [I - T^\lambda]\, V^\lambda \qquad (4.5)$$

we get (4.1).

We have seen in Section 3 that V^λ increases with λ. From now on in this paper we shall assume

$$V^\lambda \mathfrak{f} \uparrow V\mathfrak{f} \in L_m{}^q(B_a) \qquad (4.6)$$

for all $\mathfrak{f} \geqslant 0$ in $L_m{}^q(B_a)$. This corresponds to the case in [3] where the limiting matrix $[V_{ij}]$ is everywhere finite. We shall investigate the situation where $V\mathfrak{f}$ is not m-integrable in a future paper. Our assumption (4.6) implies in fact that $V^\lambda \to V$ strongly if $1 \leqslant q < \infty$, i.e., pointwise in $L_m{}^q(B_a)$—and pointwise in the w^* topology if $q = \infty$. The hypothesis (4.6) will be taken for granted from this point on without specific mention in each lemma and theorem. Note that the positivity of the everywhere defined linear operator V from L^p to itself implies its continuity.

THEOREM 4.1. *If $0 \leqslant T^\lambda \leqslant S^\lambda$ in Lemma 4.1, then*

$$\sup_\lambda \| S^\lambda \| = M \leqslant \| V \| \tag{4.7}$$

and the families $\{T^\lambda\}$, $\{S^\lambda\}$ are relatively compact in the weak operator topology if $q \geqslant 1$. Thus there exist linear operators $0 \leqslant T \leqslant S = [I - T] V$ and a net $\{\lambda_\alpha\}$ with $\lambda_\alpha \to +\infty$ such that $S^{\lambda_\alpha} \to S$, $T^{\lambda_\alpha} \to T$ in the weak operator topology and

$$[I - S]\, z^q(F) = [I - T]\, [d^0(-AF) + q(F)]. \tag{4.8}$$

Furthermore $\| T \| \leqslant \| S \| \leqslant M$. If $q = \infty$ the same is true if we replace the weak operator topology by pointwise convergence in the w^* topology.

PROOF. The only part of this theorem that is not immediate is the relative compactness of the two families of operators $\{S^\lambda\}$ and $\{T^\lambda\}$. First, if $q > 1$ then the unit ball in $\mathscr{L}(L_m{}^q, L_m{}^q)$—the space of continuous linear mappings from $L_m{}^q$ to itself—is weakly relatively compact in the weak operator topology and we are done. Hence we assume that $q = 1$. Now a subset G of E^S for E a Hausdorff linear topological space is compact in the topology of pointwise convergence on S iff G is closed, and for each $t \in S$, the closure of $\{g(t) : g \in G\}$ is compact. Now let $S = L_m{}^1(B_a)$ and $E = L_m{}^1(B_a)$ with the weak topology. We put $G =$ the closure of the set $\{S^\lambda : \lambda > 0\}$ in the topology of pointwise convergence on $L_m{}^1$, i.e. in the weak operator topology. Now for each $\mathfrak{f} \in L_m{}^1$ we have

$$| S^\lambda \mathfrak{f} | \leqslant S^\lambda | \mathfrak{f} | \leqslant V^\lambda | \mathfrak{f} | \leqslant V | \mathfrak{f} | \in L_m{}^1.$$

Thus the family $\{S^\lambda \mathfrak{f}\}$ is a uniformly integrable family in $L_m{}^1$ and is therefore relatively weakly compact by the Dunford-Pettis theorem. So the set $\{S^\lambda \mathfrak{f}; \lambda > 0\}$ has compact closure for each $\mathfrak{f} \in L_m{}^1$. Since $T^\lambda \leqslant S^\lambda$, the same arguments can be applied to $\{T^\lambda : \lambda > 0\}$. We can then choose a net $\lambda_\alpha \to \infty$ such that $T^{\lambda_\alpha} \to T$ and $S^{\lambda_\alpha} \to S$ in the weak operator topology and clearly

$$\| T \| \leqslant \| S \| \leqslant M. \tag{4.9}$$

Finally, if $q = \infty$ the proof is similar using the fact that the unit ball in L_m^∞ is w^* compact.

Condition (4.8) is called a "lateral condition" and may be thought of as a generalized boundary condition. Note that $[I - T] V \geqslant T \geqslant 0$ implies the existence of $[I - T]^{-1}$, so that (4.8) is a non-trivial condition.

5. A FORMULA FOR g^λ IN TERMS OF THE LIMITING OPERATORS

THEOREM 5.1. *If $R_\lambda \geqslant R_\lambda{}^0$ is a pseudo-resolvent of the form* (3.3)-(3.4) *with $0 \leqslant T^\lambda \leqslant S^\lambda$ then g^λ satisfies:*

$$[I - S + (I - T) V^\lambda]\, g^\lambda(f) = [I - T]\, d^\lambda(f), \tag{5.1}$$

where S and T are the limiting operators in Theorem 4.1 satisfying

$$0 \leqslant T \leqslant S. \tag{5.2}$$

If

$$[I - S]^{-1} \mathfrak{f} = \lim_{N\to\infty} \sum_{k=1}^{N} S^k \mathfrak{f} \tag{5.3}$$

exists as an $L_m{}^q$ limit then

$$[I - S + (I - T) V^\lambda]^{-1} \mathfrak{f} = \lim_{N\to\infty} \sum_{k=0}^{N} [S - (I - T) V^\lambda]^n \mathfrak{f} \tag{5.4}$$

exists in $L_m{}^q$ and (5.1) *can be written*

$$g^\lambda(f) = [I - S + (I - T) V^\lambda]^{-1} [I - T] d^\lambda(f). \tag{5.5}$$

PROOF. Putting $F = R_\lambda f$ and $g^\lambda(f) = z^a(R_\lambda f)$ in (4.8) we get

$$[I - S] g^\lambda(f) = [I - T] [d^0(-AR_\lambda f) + q(R_\lambda f)]. \tag{5.6}$$

But by (3.5) this gives

$$[I - S] g^\lambda(f) = [I - T] [d^\lambda(f) - V^\lambda g^\lambda(f)], \tag{5.7}$$

which is equivalent to (5.1). To prove (5.4) we shall need the following lemma.

LEMMA 5.1. *For each $\lambda > 0$ and each $f \geqslant 0$ in $X_m{}^q$*

$$[I - T] d^\lambda(f) \geqslant 0. \tag{5.8}$$

For any T and any admissible family $\{d^\lambda; \lambda > 0\}$, (5.8) *implies that for each* $\mathfrak{g} \geqslant 0$ *in $L_m{}^q(B_a)$*

$$0 \leqslant [I - T] V^\lambda \mathfrak{g} \uparrow S\mathfrak{g} = [I - T] V\mathfrak{g}. \tag{5.9}$$

PROOF. Suppose $f \geqslant 0$ and $\mu \geqslant \lambda > 0$. Then by (2.22)

$$\begin{aligned}[I - T^\mu] d^\lambda(f) &= [I - T^\mu] d^\mu(f) + (\mu - \lambda) [I - T^\mu] d^\mu(R_\lambda{}^0 f) \\ &= g^\mu[f + (\mu - \lambda) R_\lambda{}^0 f] \geqslant 0. \end{aligned} \tag{5.10}$$

Therefore, letting $\mu \to \infty$ through a suitable net gives (5.8). The operator $[I - T] V^\lambda$ is positive for $\lambda > 0$ by (5.8) and (3.2). Also if $\lambda \geqslant \mu$ then

$$[I - T] [V^\lambda - V^\mu] \mathfrak{g} = (\lambda - \mu) [I - T] d^\lambda(h^\mu(x_\mathfrak{g})) \geqslant 0. \tag{5.11}$$

Since $V^\lambda \to V$ strongly in $L_m{}^q$ if $1 \leqslant q < \infty$, and in the w^* operator topology if $q = \infty$ we have (5.9). This completes the proof of the lemma.

Now returning to the proof of the theorem we see that (5.9) implies

$$0 \leqslant [S - (I - T) V^\lambda] \leqslant S \tag{5.12}$$

and so (5.4) clearly holds for $f \geqslant 0$. But this implies that it holds for all f.

COROLLARY 5.1. *Under the hypotheses of Theorem 5.1 we have*

$$S^\lambda = [I - S + (I - T) V^\lambda]^{-1} [I - T] V^\lambda \tag{5.13}$$

if (5.3) *holds.*

PROOF. This follows from (2.8) and (5.5).

Since $S^\lambda = [I - T^\lambda] V^\lambda$, it is natural to ask, in view of (5.5), under what conditions we have

$$[I - T^\lambda] = [I - S + (I - T) V^\lambda]^{-1} [I - T]. \tag{5.14}$$

If T^λ is defined by (5.14) without further restrictions it will not necessarily be positive.

LEMMA 5.2. *Under the conditions of Theorem 1.1 with* $\| S \| < 1$, *we have* (5.14) *if and only if there exists a positive continuous operator*

$$P : L_m{}^q(B_a) \to L_m{}^q(B_a)$$

such that

$$[I - T^\lambda]^{-1} [I - S^\lambda] = I - P \qquad \forall\lambda > 0. \tag{5.15}$$

If such a P exists, then also

$$[I - T]^{-1} [I - S] = I - P. \tag{5.16}$$

PROOF. First suppose that (5.14) holds. From (5.1) and the definition of S we have

$$[I - S^\lambda] = [I - S + (I - T) V^\lambda]^{-1} [I - S]. \tag{5.17}$$

Under the hypotheses of Theorem 5.1 if (5.3) holds for S it also holds for T and so $[I - T]^{-1}$ exists. Hence, combining (5.14) and (5.17) we get

$$\begin{aligned}[I - T^\lambda]^{-1} [I - S^\lambda] &= [I - T]^{-1} [I - S] \\ &= I - [I - T]^{-1} [S - T]. \end{aligned} \tag{5.18}$$

If we put

$$P = [I - T]^{-1} [S - T] \tag{5.19}$$

we have $P \geqslant 0$ by (5.2).

Now for the sufficiency. Clearly if P is given by (5.15) independent of λ we have

$$[I - S^\lambda] = [I - T^\lambda] [I - P].$$

Then letting $\lambda \to \infty$ through an appropriate net we get (5.16). Thus

$$\begin{aligned}[I - T^\lambda] &= [I - S^\lambda] [I - P]^{-1} \\ &= [I - S^\lambda] [I - S]^{-1} [I - T]. \end{aligned}$$

But we know that $I - S^\lambda$ is given by (5.17) and so (5.14) must hold.

We use the result of Lemma 5.2 now to show that the particular representation with T^λ given by (2.9) and d^λ by (2.13) does satisfy the condition of Lemma 5.2. By Corollary 2.1 this representation will certainly exist if $[I - S^\lambda]^{-1}$ exists for all $\lambda > 0$. Note that there is no nonuniqueness involved in the choice of S^λ, since that operator is defined unambiguously in terms of g^λ. The freedom of choice lies in T^λ and d^λ.

PROPOSITION 5.1. *If $R_\lambda \geqslant R_\lambda{}^0$ is a pseudo-resolvent of the form* (3.3)-(3.4) *with*

$$[I - S^\lambda]^{-1}\,\mathfrak{f} = \lim_{N\to\infty} \sum_{k=0}^{N} (S^\lambda)^n \mathfrak{f} \tag{5.20}$$

existing as an $L_m{}^q$ limit for each $\mathfrak{f} \in L_m{}^q$, if T^λ is given by (2.9) *and d^λ by* (2.13) *then* (5.15) *holds with*

$$P\mathfrak{f} = \lim_{\mu\downarrow 0} V^\mu \mathfrak{f}. \tag{5.21}$$

Thus T^λ satisfies (5.14).

PROOF. Since $g^\lambda = [I - T^\lambda]\, d^\lambda$ we have

$$T^\lambda \mathfrak{f} = \lim_{\mu\downarrow 0} g^\lambda[h^\mu(x_\mathfrak{f})] = \lim_{\mu\downarrow 0} [I - T^\lambda]\, d^\lambda[h^\mu(x_\mathfrak{f})] = [I - T^\lambda]\, W^\lambda \mathfrak{f}, \tag{5.22}$$

where

$$W^\lambda \mathfrak{f} = \lim_{\mu\downarrow 0} d^\lambda[h^\mu(x_\mathfrak{f})]. \tag{5.23}$$

This limit exists for each $\mathfrak{f}$. It is sufficient to prove it for $\mathfrak{f} \geqslant 0$. The mapping $\mathfrak{f} \to d^\lambda[h^\mu(x_\mathfrak{f})]$ is positive for each λ, $\mu > 0$ and for fixed λ increases as $\mu \downarrow 0$ in the usual ordering for operators. Furthermore,

$$d^\lambda[h^\mu(x_\mathfrak{f})] \leqslant d^\lambda(x_\mathfrak{f}).$$

Therefore the limit in (5.23) exists for each $\mathfrak{f} \in L_m{}^q(B_a)$ and $W^\lambda \mathfrak{f} \in L_m{}^q(B_a)$.

We next show that

$$V^\lambda \mathfrak{f} - W^\lambda \mathfrak{f} = \lim_{\mu\downarrow 0} V^\mu \mathfrak{f} = P\mathfrak{f} \qquad \forall \lambda > 0. \tag{5.24}$$

We have

$$V^\lambda \mathfrak{f} - W^\lambda \mathfrak{f} = \lim_{\mu\downarrow 0} \lambda\, d^\lambda[x_\mathfrak{f} - h^\mu(x_\mathfrak{f})] = \lim_{\mu\downarrow 0} \lambda\, d^\lambda[\mu R_\mu{}^0 x_\mathfrak{f}]. \tag{5.25}$$

But using (2.22)

$$\begin{aligned} \lim_{\mu\downarrow 0} \lambda\, d^\lambda[\mu R_\mu{}^0 x_\mathfrak{f}] &= \lim_{\mu\downarrow 0} (\lambda\mu)\,(\mu - \lambda)^{-1}\,[d^\lambda(x_\mathfrak{f}) - d^\mu(x_\mathfrak{f})] \\ &= \lim_{\mu\downarrow 0} V^\mu \mathfrak{f} = P\mathfrak{f}. \end{aligned} \tag{5.26}$$

Hence, by (5.22)

$$T^\lambda = [I - T^\lambda]\,[V^\lambda - P] = S^\lambda - [I - T^\lambda]\,P, \tag{5.27}$$

or equivalently

$$[I - T^\lambda]\,[I - P] = [I - S^\lambda]. \tag{5.28}$$

Since $0 \leqslant T^\lambda \leqslant S^\lambda$ we know that $[I - T^\lambda]^{-1}$ exists and (5.28) is then equivalent to (5.15).

6. Pseudo-Resolvents Satisfying Given Lateral Conditions

In the next two sections we turn to the converse of the problem studied in the first part of the paper. In Theorem 4.1 we started with a given pseudo-resolvent and showed that the elements in its range satisfy the lateral condition (4.8). In this section we start with a lateral condition of the type (4.8) and then construct a pseudo-resolvent whose range is defined by the given lateral condition, cf. Theorem 6.1.

Throughout Sections 6 and 7 the following will be assumed. There are given a continuous, positive linear operator from $L_m{}^q(B_a)$ into itself and an admissible family $\{d^\lambda; \lambda > 0\}$ of positive continuous linear mappings from $X_m{}^q$ into $L_m{}^q(B_a)$ such that

$$[I - T]\,d^\lambda(f) \geqslant 0 \qquad \forall f \in (X_m{}^q)^+. \tag{6.1}$$

As in the first part of the paper the operator V^λ defined by (3.2) will be subject to the condition (4.6). Now define S by

$$S = [I - T]\,V \tag{6.2}$$

and assume

$$[I - S]^{-1}\mathfrak{f} = \lim_{N\to\infty} \sum_{n=0}^{N} S^n \mathfrak{f} \tag{6.3}$$

exists as an $L_m{}^q(B_a)$ limit for each $\mathfrak{f} \in L_m{}^q(B_a)$. Associated with our given family of admissible mappings we have the transformations d^0 and q defined by (3.5), (3.1), and (4.2).

THEOREM 6.1. *Define*

$$R_\lambda f = R_\lambda{}^0 f + h^\lambda \left\{ \int_{B_a} g^\lambda(f, y)\, y m(dy) \right\}, \tag{6.4}$$

where

$$g^\lambda(f) = [I - S + (I - T)\,V^\lambda]^{-1}\,[I - T]\,d^\lambda(f). \tag{6.5}$$

Then R_λ is a pseudo-resolvent, $R_\lambda \geqslant R_\lambda^0$, and each $F \in$ range R_λ satisfies

$$[I - S]\, z^a(F) = [I - T]\, [d^0(-AF) + q(F)]. \tag{6.6}$$

PROOF. First hold λ and f fixed and let $F = R_\lambda f$. Then

$$z^a(F) = g^\lambda(f) = [I - S + (I - T)\, V^\lambda]^{-1}\, [I - T]\, d^\lambda(\lambda F - AF). \tag{6.7}$$

Note that the inverse in (6.5) exists since

$$S \geqslant [I - T]\, V^\lambda \geqslant 0 \tag{6.8}$$

by (6.1) and Lemma 5.1, cf. (5.9). Then (6.3) implies (5.4). Now from (6.7)

$$[I - S]\, z^a(F) = [I - T]\, [d^\lambda(\lambda F - AF) - V^\lambda z^a(F)]. \tag{6.9}$$

By Proposition 3.1

$$d^\lambda(\lambda F - AF) - V^\lambda z^a(F) = d^0(-AF) + q(F), \tag{6.10}$$

so (6.6) holds for $F = R_\lambda f$. We have then a solution $F = R_\lambda f$ to the equation $\lambda u - Au = f$ subject to the lateral condition (6.6). Let us next show that there can be at most one such solution. If there existed two such solutions F_1 and F_2 in $X_m{}^q$ then the difference $u = F_1 - F_2$ would be in $N_m{}^\lambda$ and therefore be of the form

$$u = h^\lambda(x_g) \tag{6.11}$$

for some $g \in L_m{}^q(B_a)$. Then since u would satisfy the lateral condition (6.6),

$$[I - S]\, z^a(u) = [I - S]\, g = [I - T]\, [d^0(-\lambda u) + q(u)]. \tag{6.12}$$

But $u = -AR_\lambda{}^0 x_g$, and so it follows from the original definition of d^0, cf. (3.6), that

$$\lambda\, d^0(u) = \lambda\, d^0(-AR_\lambda{}^0 x_g) = \lambda\, d^\lambda(x_g) + q(u). \tag{6.13}$$

Thus

$$d^0(-\lambda u) + q(u) = -\lambda\, d^\lambda(x_g) = -V^\lambda g. \tag{6.14}$$

But substituting this in (6.12) would give

$$[I - S + (I - T)\, V^\lambda]\, g = 0, \tag{6.15}$$

and this in turn implies $g = 0$ because of (5.4). This uniqueness implies that R_λ is a pseudo-resolvent. The condition $R_\lambda \geqslant R_\lambda{}^0$ is assured by (6.1) and (6.8).

7. Construction of Semigroups from Given Lateral Conditions

In this section we find conditions under which the pseudo-resolvents constructed in Theorem 6.1 are actually resolvents of equicontinuous semigroups of class C_0 on subspaces of $X_m{}^q$. We refer to Yosida [9] pp. 234-250 for the main facts about equicontinuous semigroups of class C_0 in locally convex spaces. We shall have to impose more restrictions now on the given spaces. We assume (i) The positive cone is normal. In our case this is equivalent to the condition that the topology is determined by a family $\mathscr{P}$ of monotone seminorms, i.e., $p \in \mathscr{P}$ implies that $p(x) \leqslant p(y)$ whenever $0 \leqslant x \leqslant y$. We refer to [7] p. 215 for further properties of normal cones. For example, it is easily seen that if the cone is normal there exists a locally compact space T such that X is isomorphic to a subspace of $R(T)$, the space of continuous, real-valued functions on T with the topology of compact convergence. We shall not make use of this representation in the sequel, however.

In addition to this we assume that (ii) X_0 is sequentially complete and barrelled. Then the same is true for $X_m{}^q$, since it is the topological direct sum of X_0 and the Banach space $N_m{}^a$ with the topology induced by $X_m{}^q$. Condition (ii) is satisfied if, for example, X_0 is a Banach space or, more generally, a complete metrizable topological vector space.

The assumptions made at the beginning of Section 6, including (6.1)-(6.3), will remain in force throughout this section.

LEMMA 7.1. *Suppose* $\mathfrak{v} \geqslant 0$ *belongs to* $L_m{}^q(B)$. *Let*

$$\mathfrak{v}_a = \mathfrak{v}/B_a \qquad \textit{and} \qquad \mathfrak{v}_p = \mathfrak{v}/B_p$$

If R_λ *is the pseudo-resolvent defined by* (6.4) *and* (6.5) *in Theorem 6.1 and* $x_{\mathfrak{v}}$ *is defined as in* (2.7), *then*

$$\lambda R_\lambda x_{\mathfrak{v}} \leqslant x_{\mathfrak{v}} \tag{7.1}$$

if and only if

$$[I - S]\, \mathfrak{v}_a \geqslant [I - T]\, \tau(\mathfrak{v}_p). \tag{7.2}$$

PROOF. We have

$$\lambda R_\lambda x_{\mathfrak{v}} - x_{\mathfrak{v}} = h^\lambda \left\{ \int_{B_a} [\lambda g^\lambda(x_{\mathfrak{v}}; y) - \mathfrak{v}_a(y)]\, ym(dy) \right\}. \tag{7.3}$$

Hence $\lambda R_\lambda x_{\mathfrak{v}} \leqslant x_{\mathfrak{v}}$ if and only if

$$\lambda g^\lambda(x_{\mathfrak{v}}) \leqslant \mathfrak{v}_a\,. \tag{7.4}$$

But

$$\lambda g^\lambda(x_{\mathfrak{v}}) = \lambda g^\lambda(x_{\mathfrak{v}_a}) + \lambda g^\lambda(x_{\mathfrak{v}_p}) = S^\lambda \mathfrak{v}_a + [I - T^\lambda]\, \tau(\mathfrak{v}_p). \tag{7.5}$$

LEMMA 7.2. *Suppose*[1] *that* $f_0 \in X_m{}^q$ *satisfies*

$$0 \leqslant \lambda R_\lambda{}^0 f_0 \leqslant f_0 \qquad \forall \lambda > 0. \tag{7.6}$$

[1] There is an error in [3] Lemma 1.4 as it stands. Lemma 7.2 of the present paper gives the corrected version.

Define $\mathfrak{u} \in L_m{}^q(B_a)$ *by*

$$[I - S]\,\mathfrak{u} = [I - T]\,d^\nu(f_0) \tag{7.7}$$

for some $\nu > 0$, *and let*

$$R_\nu{}^0 f_0 + x_{\mathfrak{u}} = g. \tag{7.8}$$

then

$$0 \leqslant \lambda R_\lambda g \leqslant g \qquad \forall \lambda > 0 \tag{7.9}$$

where R_λ *is the pseudo-resolvent in Theorem 6.1.*

PROOF. First

$$0 \leqslant \lambda R_\lambda R_\nu{}^0 f_0 = \lambda R_\lambda{}^0 R_\nu{}^0 f_0 + h^\lambda \left\{\int_{B_a} \lambda g^\lambda(R_\nu{}^0 f_0; y)\, y m(dy)\right\}$$

$$\leqslant R_\nu{}^0 f_0 + h^\lambda \left\{\int_{B_a} \lambda g^\lambda(R_\nu{}^0 f_0; y)\, y m(dy)\right\}. \tag{7.10}$$

Further,

$$\lambda R_\lambda x_{\mathfrak{u}} = x_{\mathfrak{u}} + h^\lambda \left[\int_{B_a} \{\lambda g^\lambda(x_{\mathfrak{u}}; y) - \mathfrak{u}(y)\}\, y m(dy)\right], \tag{7.11}$$

bearing in mind that $h^\lambda(x_p) = 0$ for any passive solution x_p. So

$$\lambda R_\lambda g \leqslant g + h^\lambda \left[\int_{B_a} \{\lambda g^\lambda(R_\nu{}^0 f_0; y) + \lambda g^\lambda(x_{\mathfrak{u}}; y) - \mathfrak{u}(y)\}\, y m(dy)\right]. \tag{7.12}$$

Now

$$\lambda g^\lambda(R_\nu{}^0 f_0) + \lambda g^\lambda(x_{\mathfrak{u}}) - \mathfrak{u} = [I - S + (I - T)\,V^\lambda]^{-1}\,[I - T]\,\lambda\, d^\lambda(R_\nu{}^0 f_0) + S^\lambda \mathfrak{u} - \mathfrak{u}$$

$$= [I - S + (I - T)\,V^\lambda]^{-1}\,[(I - T)\,\lambda\, d^\lambda(R_\nu{}^0 f_0) - (I - S)\,\mathfrak{u}], \tag{7.13}$$

using (5.17). Finally by (7.7) and (2.22)

$$[I - T]\,\lambda\, d^\lambda(R_\nu{}^0 f_0) - [I - S]\,\mathfrak{u} = [I - T]\,[\lambda\, d^\lambda(R_\nu{}^0 f_0) - d^\nu(f_0)]$$

$$= [I - T]\,d^\lambda(\nu R_\nu{}^0 f_0 - f_0) \leqslant 0. \tag{7.14}$$

Since $I - S + (I - T)\,V^\lambda$ has a positive inverse we conclude that

$$\lambda g^\lambda(R_\nu{}^0 f_0) + \lambda g^\lambda(x_{\mathfrak{u}}) - \mathfrak{u} \leqslant 0 \tag{7.15}$$

and so the second term on the right in (7.12) is nonpositive, which proves (7.9).

In the rest of this section the following domination condition will play an important role:

(D) Given $f \geqslant 0$ in $X_m{}^q$ there exists an f_0 in $X_m{}^q$ satisfying (7.6) and an $x_\mathfrak{v} \in [N_m{}^q]^+$ satisfying (7.2) such that for some $\nu > 0$

$$f \leqslant R_\nu{}^0 f_0 + x_\mathfrak{v} \,. \tag{7.16}$$

LEMMA 7.3. *If condition* (D) *holds then, given $f \geqslant 0$ in $X_m{}^q$ there exists a $k \in X_m{}^q$ such that*

$$0 \leqslant \lambda R_\lambda k \leqslant k \tag{7.17}$$

and

$$f \leqslant k. \tag{7.18}$$

PROOF. We choose f_0 and $x_\mathfrak{v}$ so that (7.16) holds. By Lemma 7.2 if we define g by (7.8) then (7.9) holds. Also by Lemma 7.1 we know that (7.1) holds. Then if $k = g + x_\mathfrak{v}$, we have both (7.17) and (7.18).

We introduce some preliminary notation before proceeding to the main theorem of the section. Let, for a given pseudo-resolvent R_λ,

$$Y_m{}^q = \overline{R_\lambda X_m{}^q} \tag{7.19}$$

and

$$\Sigma_m{}^q = \{y \in Y_m{}^q : Ay \in Y_m{}^q \text{ and } [I - S]\, z^a(y) = [I - T]\, [d^0(-Ay) + q(y)]\}. \tag{7.20}$$

Then define

$$A_m{}^q = A / \Sigma_m{}^q \tag{7.21}$$

THEOREM 7.1. *If R_λ is the pseudo-resolvent of Theorem 6.1 defined by* (6.4) *and* (6.5), *and if condition* (D) *holds, then the family $\{\lambda^n R_\lambda{}^n : \lambda > 0, n \in N\}$ of continuous linear transformations is an equicontinuous family on $X_m{}^q$. There exists a semigroup $\{T_t^{(m,q)}\}$, equicontinuous of class C_0 on $Y_m{}^q$, whose resolvent is R_λ and whose infinitesimal generator is $A_m{}^q$.*

PROOF. First let us show that $\Sigma_m{}^q = \text{range } R_\lambda$. By Theorem 6.1 we have range $R_\lambda \subset \Sigma_m{}^q$. On the other hand, if $y \in \Sigma_m{}^q$ then

$$\lambda y - A_m{}^q y = f \in Y_m{}^q$$

and so $y = R_\lambda f$, since we showed in the proof of Theorem 6.1 that $R_\lambda f$ is the unique solution of this equation in $X_m{}^q$. Thus

$$\Sigma_m{}^q = \text{range } R_\lambda = \text{domain } A_m{}^q.$$

But this implies that domain $A_m{}^q$ is dense in $Y_m{}^q$, and also

$$R_\lambda = (\lambda I - A_m{}^q)^{-1}.$$

The equicontinuity of the family $\{\lambda^n R_\lambda{}^n; \lambda > 0, n \in N\}$ is proved as follows: if $f \in X_m{}^q$ then $f = f^+ - f^-$ with f^+ and f^- both in $[X_m{}^q]^+$. Choose k_1 and k_2 to satisfy

$$f^+ \leqslant k_1, \qquad f^- \leqslant k_2, \qquad 0 \leqslant \lambda R_\lambda k_i \leqslant k_i, \tag{7.22}$$

which is possible by Lemma 7.3. Then if $p' \in \mathscr{P}'(q)$, we have

$$\begin{aligned} p'(\lambda^n R_\lambda{}^n f) &\leqslant p'(\lambda^n R_\lambda{}^n f^+) + p'(\lambda^n R_\lambda{}^n f^-) \\ &\leqslant \sum_{i=1}^{2} p'(\lambda^n R_\lambda{}^n k_i) \leqslant \sum_{i=1}^{2} p'(k_i). \end{aligned} \tag{7.23}$$

Thus the family $\{\lambda^n R_\lambda{}^n; \lambda > 0, n \in N\}$ is pointwise bounded and thus equicontinuous on $X_m{}^q$ (and hence on $Y_m{}^q$), since $X_m{}^q$ is barrelled.

The second part of the theorem is now an immediate consequence of the Hille-Yosida theorem for locally convex spaces, cf. [9] p. 246.

The method used to prove Theorem 7.1 is most simply illustrated when the positive cone is normal. However, this normality condition can be considerably weakened. The same sort of proof goes through with only minor modifications, for example, if the topology on $X_m{}^q$ is generated by a family of seminorms $\mathscr{P}'$ which are monotone on $N_m{}^q$, but on range $R_\lambda{}^0$ satisfy the weaker condition that

$$p'(R_\lambda{}^0 f) \leqslant q_1(R_\lambda{}^0 f) \cdot q_2(f)$$

where the q_i are monotone seminorms. We shall discuss such modifications in Part II, where specific applications and examples will be presented.

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